Covariance :

Let X & Y be two random variables

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Correlation Coefficient (Karl Pearson's Coefficient of Correlation)

$$\gamma_{xy} = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{E(X^2) - (E(X))^2} \; ; \; \sigma_y = \sqrt{E(Y^2) - (E(Y))^2}$$

Note: If X and Y are independent random variables then X & Y are uncorrelated.

Proof: WKT for independent random variables

$$E(XY) = E(X)E(Y)$$

$$\therefore \; Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$Cov(X,Y) = 0$$

Hence $\gamma_{xy} = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = 0$

$\Rightarrow$ X and Y are uncorrelated.

But the converse need not be true.

(ie) uncorrelated random variables need not be independent

For ex:

X :	-2	-1	0	1	2
$Y = X^2$:	4	1	0	1	4

$$E(X) = \frac{\Sigma x}{n} = \frac{-2-1+0+1+2}{5} = 0$$

$$E(XY) = \frac{-8-1+1+8}{5} = 0$$

$$\therefore \; \gamma_{xy} = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sigma_x \sigma_y} = 0$$

X & Y are uncorrelated. But $Y = X^2$

They are dependent.

Hence uncorrelated rvs need not be independent.

Properties :

1. $Cov(aX, bY) = ab\, cov(X, Y)$

2. $Cov(X+a, Y+b) = Cov(X, Y)$

3. $Cov(aX+b, cY+d) = ac\, cov(X, Y)$

4. If X and Y are independent then $Cov(X, Y) = 0$.

5. $V(X_1 \pm X_2) = V(X_1) + V(X_2) \pm 2\,Cov(X_1, X_2)$ if X_1 & X_2 are not indept.

Result: P.T Correlation Coefficient lies b/w -1 and 1

i.e, T.P.T $-1 \leq r_{xy} \leq 1$.

Soln:

Let $E[a(x-\bar{x}) + (Y-\bar{Y})]^2 \geq 0$

$$a^2 E[x-\bar{x}]^2 + E(Y-\bar{Y})^2 + 2aE(x-\bar{x})(Y-\bar{Y}) \geq 0$$

$$a^2 \sigma_x^2 + \sigma_y^2 + 2a\,Cov(x, Y) \geq 0$$

The above eqn ≥ 0 only when $B^2 - 4AC \not> 0$

i.e, $(2\,cov(X, Y))^2 - 4\sigma_x^2 \sigma_y^2 \not> 0$

$$\cancel{4}(Cov(x, Y))^2 \leq \cancel{4}\sigma_x^2 \sigma_y^2$$

$$\frac{c_{xy}^2}{\sigma_x^2 \sigma_y^2} \leq 1$$

$$r_{xy}^2 \leq 1$$

$|r_{xy}| \leq 1$ or $-1 \leq r_{xy} \leq 1$.

Formula:

Discrete Random Variable

$E(X) = \sum_i x_i P_{i*}$

$E(Y) = \sum_j y_j P_{*j}$

$E(X^2) = \sum x_i^2 P_{i*}$

$E(Y^2) = \sum y_j^2 P_{*j}$

$E(XY) = \sum_j \sum_i x_i y_j P_{ij}$

Continuous Random Variable

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy$

$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy$

1. Let X and Y be discrete RVs' with probability function $f(x,y) = \frac{x+y}{21}$, $x = 1,2,3$; $y = 1,2$. Find (i) Mean and variance of X and Y (ii) $Cov(X,Y)$ (iii) Correlation of X and Y.

Soln:

Y \ X	1	2	3	$P\{Y=y_j\}$
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{12}{21}$
$P\{X=x_i\}$	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	$\frac{21}{21}=1$

Marginal dist of X:

X	1	2	3	Total
P	5/21	7/21	9/21	1

Marginal distribution of Y

Y	1	2	T
$P\{Y=y\}$	9/21	12/21	1

$E(X) = \sum x_i P_i$

$= 1\cdot\frac{5}{21} + 2\cdot\frac{7}{21} + 3\cdot\frac{9}{21}$

$= \frac{46}{21}$

$E(Y) = \sum y_j P_j$

$= 1\cdot\frac{9}{21} + 2\cdot\frac{12}{21}$

$E(Y) = \frac{33}{21}$

$$E(x^2) = \sum x_i^2 p_i$$

$$= 1 \cdot \frac{5}{21} + 4\left(\frac{7}{21}\right) + 9\left(\frac{9}{21}\right)$$

$$= \frac{114}{21}$$

$$E(y^2) = \sum y_j^2 p_j = \frac{9}{21} + 4\left(\frac{12}{21}\right)$$

$$= \frac{57}{21}$$

$$E(xy) = \sum x_i y_j p_{ij} = 1.1\left(\frac{2}{21}\right) + 2.1\left(\frac{3}{21}\right) + 3.1.\left(\frac{4}{21}\right) +$$
$$1.2.\left(\frac{3}{21}\right) + 2.2.\left(\frac{4}{21}\right) + 3.2.\left(\frac{5}{21}\right)$$

$$E(xy) = \frac{72}{21}$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{114}{21} - \left(\frac{46}{21}\right)^2 = \frac{278}{441}$$

$$V(y) = E(y^2) - (E(y))^2 = \frac{57}{21} - \left(\frac{33}{21}\right)^2 = \frac{12}{49}$$

$$Cov(x, y) = E(xy) - E(x)E(y)$$

$$= \frac{72}{21} - \left(\frac{46}{21}\right)\left(\frac{33}{21}\right) = \cancel{1.508}\ 56 \quad (3.428) - (2.19)(1.57) = -0.0136$$

$$r_{xy} = \frac{Cov(x, y)}{\sigma_x \sigma_y} = \frac{\cancel{1508.56}\ -0.01436}{\sqrt{\frac{278}{441}}\sqrt{\frac{12}{49}}}$$

$$= \frac{-0.0103}{(0.794)(0.495)} = \frac{-0.01436}{0.393}$$

$$= -0.0346$$

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2. Two random rvs X & Y have the joint density

$$f(x,y) = \begin{cases} 2-x-y, & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{S.T } cov(X,Y) = -\frac{1}{144}$$

Also find r_{xy}.

Soln:

$$Cov(X,Y) = E(XY) - E(X)E(Y).$$

Let $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (2-x-y)\,dy = \left(2y - xy - \frac{y^2}{2}\right)_0^1$

$$f(x) = \left(2 - x - \frac{1}{2} - 0\right) = \frac{3}{2} - x$$

$$\boxed{f(x) = \frac{3}{2} - x, \quad 0 \le x \le 1}$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (2-x-y)\,dx$$

$$= \left(2x - \frac{x^2}{2} - xy\right)_0^1$$

$$= \left(2 - \frac{1}{2} - y - 0\right)$$

$$= \left(\frac{3}{2} - y\right)$$

$$\boxed{f(y) = \frac{3}{2} - y, \quad 0 \le y \le 1}$$

Let $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \int_0^1 \left(\frac{3}{2}x - x^2\right)dx$

$$= \left(\frac{3}{2}\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \left(\frac{3}{4} - \frac{1}{3} - 0\right)$$

$$E(X) = \frac{5}{12}$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y\left(\frac{3}{2} - y\right)dy = \int_0^1 \left(\frac{3}{2}y - y^2\right)dy$$

$$E(Y) = \frac{5}{12}$$

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$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx$$

$$= \left(\frac{3}{2}\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\boxed{E(x^2) = \frac{1}{4}}$$

$$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 y^2\left(\frac{3}{2} - y\right)dy$$

$$= \left(\frac{3}{2}\frac{y^3}{3} - \frac{y^4}{4}\right)_0^1 = \frac{1}{4}$$

$$\boxed{E(y^2) = \frac{1}{4}}$$

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy\,(2 - x - y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (2xy - x^2y - xy^2)\,dx\,dy$$

$$= \int_0^1 \left(\frac{\cancel{2}x^2y}{\cancel{2}} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right)_0^1 dy$$

$$= \int_0^1 \left[\left(y - \frac{y}{3} - \frac{y^2}{2}\right) - 0\right] dy$$

$$= \left(\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6}\right)_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{6} - 0\right) \qquad \frac{1}{2} - \frac{2}{6}$$

$$E(xy) = \frac{1}{6}$$

$$\text{cov}(x,y) = E(xy) - E(x)E(y)$$

$$= \frac{1}{6} - \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{-1}{144}$$

$$r_{xy} = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$$

$$V(X) = E(x^2) - (E(x))^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144}$$

$$V(Y) = E(y^2) - (E(y))^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144}$$

$$r_{xy} = \frac{-1/144}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}}$$

$$r_{xy} = \frac{-1/144}{\frac{11}{144}} = \frac{-1}{11}.$$

3. The random variable (x,y) has a joint PDF

$$f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

compute $r(x,y)$. Also check X & Y are independent or not.

Soln:

$$r_{xy} = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(x^2) - (E(x))^2}\sqrt{E(y^2) - (E(y))^2}}$$

Let $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)_0^1$

$$\boxed{f(x) = x + \frac{1}{2},\quad 0<x<1}$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx = \left(\frac{x^2}{2} + xy\right)_0^1$$

$$\boxed{f(y) = \frac{1}{2} + y,\quad 0<y<1}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(x+\frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)_0^1 = \left(\frac{1}{3} + \frac{1}{4} - 0\right)$$

$$\boxed{E(X) = \frac{7}{12}}$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y\left(y+\frac{1}{2}\right)dy$$

$$= \int_0^1 \left(y^2 + \frac{1}{2}y\right)dy = \left(\frac{y^3}{3} + \frac{y^2}{4}\right)_0^1 = \frac{7}{12}$$

$$\boxed{E(Y) = \frac{7}{12}}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 xy\,(x+y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 (x^2y + xy^2)\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^3y}{3} + \frac{x^2y^2}{2}\right)_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right)dy$$

$$= \left(\frac{y^2}{6} + \frac{y^3}{6}\right)_0^1 = \left(\frac{1}{6} + \frac{1}{6} - 0\right)$$

$$\boxed{E(XY) = \frac{1}{3}}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 x^2\left(x+\frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2}\right) dx$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{6}\right)_0^1 = \frac{1}{4} + \frac{1}{6}$$

$$\boxed{E(x^2) = \frac{5}{12}}$$

$$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)\, dy = \int_0^1 y^2 \left(y + \frac{1}{2}\right) dy$$

$$= \int_0^1 \left(y^3 + \frac{1}{2} y^2\right) dy$$

$$= \left(\frac{y^4}{4} + \frac{y^3}{6}\right)_0^1$$

$$E(y^2) = \frac{1}{4} + \frac{1}{6}$$

$$\boxed{E(y^2) = \frac{5}{12}}$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2$$

$$\boxed{Var(x) = \frac{11}{144}} \quad \boxed{\sigma_x = \frac{\sqrt{11}}{12}}$$

$$Var(y) = E(y^2) - (E(y))^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

$$\boxed{\sigma_y = \frac{\sqrt{11}}{12}}$$

$$Cov(x, y) = E(xy) - E(x)E(y)$$

$$= \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right)$$

$$= \frac{-1}{144}$$

Correlation Coefficient $r_{xy} = \dfrac{\text{cov}(x,y)}{\sigma_x \sigma_y}$

$$= \frac{-\frac{1}{144}}{\frac{\sqrt{11}}{12}\ \frac{\sqrt{11}}{12}}$$

$$= \frac{-1}{11} = -0.090$$

Here $\text{cov}(x,y) \neq 0$

$\therefore$ X & Y are not independent.

4. Given that X and Y are independent random variables which is given by $f(x) = \begin{cases} 4ax & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$, $f(y) = \begin{cases} 4by, & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

P.T ~~U & V are uncorrelated~~ $U = X+Y$, $V = X-Y$ are uncorrelated.

Soln:

u and v are uncorrelated $\Rightarrow$ $\text{cov}(u,v) = 0$

Let $\text{Cov}(u,v) = E(uv) - E(u)E(v)$

$$= E((x+y)(x-y)) - E(x+y)\,E(x-y)$$

$$= E(x^2 - y^2) - (E(x)+E(y))(E(x)-E(y))$$

$$= E(x^2) - E(y^2) - ((E(x))^2 - (E(y))^2).$$

To find a:

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 4ax\,dx = 1$$

$$\overset{2}{\cancel{4}}a\left[\frac{x^2}{\cancel{2}}\right]_0^1 = 1$$

$$2a(1-0) = 1$$

$$2a = 1 \Rightarrow \boxed{a = \tfrac{1}{2}}$$

To find b:

$$\int_{-\infty}^{\infty} f(y)\,dy = 1$$

$$\int_0^1 4by\,dy = 1$$

$$\overset{2}{\cancel{4}}b\left(\frac{y^2}{\cancel{2}}\right)_0^1 = 1$$

$$2b = 1$$

$$\boxed{b = \tfrac{1}{2}}$$

$$f(x) = \begin{cases} 4\left(\frac{1}{2}\right)x, & 0<x<1 \\ 0 & \text{otherewise} \end{cases}$$

$$f(x) = \begin{cases} 2x, & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

IIIly $$f(y) = \begin{cases} 2y, & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Let $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(2x)\,dx = 2\int_0^1 x^2\,dx$

$E(x) = 2\left[\frac{x^3}{3}\right]_0^1$

$\boxed{E(X) = \frac{2}{3}}$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2(2x)\,dx = 2\int_0^1 x^3\,dx$

$E(x^2) = 2\left(\frac{x^4}{4}\right)_0^1 = \frac{2}{4} = \frac{1}{2}$

$\boxed{E(x^2) = \frac{1}{2}}$

Let $E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y(2y)\,dy = 2\left(\frac{y^3}{3}\right)_0^1 = \frac{2}{3}$

$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 y^2(2y)\,dy = 2\left(\frac{y^4}{4}\right)_0^1 = \frac{1}{2}$

$\therefore \; Cov(u,v) = \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2\right)$

$= 0$

Let $r_{uv} = \frac{Cov(u,v)}{\sigma_u \sigma_v} = 0$

$\Rightarrow r_{uv} = 0$ ' Hence u & v are uncorrelated.

Calculate the coefficient of correlation from the following data.

x :	65	66	67	67	68	69	70	72
y :	67	68	65	68	72	72	69	71

Soln:

x	y	x^2	y^2	xy
65	67	4225	4489	4355
66	68	4356	4624	4488
67	65	4489	4225	4355
67	68	4489	4624	4556
68	72	4624	5184	4896
69	72	4761	5184	4968
70	69	4900	4761	4830
72	71	5184	5041	5112
$\Sigma x = 544$	$\Sigma y = 552$	$\Sigma x^2 = 37028$	$\Sigma y^2 = 38132$	$\Sigma xy = 37560$

$$E(x) = \frac{\Sigma x}{n} = \frac{544}{8} = 68$$

$$E(y) = \frac{\Sigma y}{n} = \frac{552}{8} = 69$$

$$E(x^2) = \frac{\Sigma x^2}{n} = \frac{37028}{8} = 4628.5$$

$$E(y^2) = \frac{\Sigma y^2}{n} = \frac{38132}{8} = 4766.5$$

$$E(xy) = \frac{\Sigma xy}{n} = \frac{37560}{8} = 4695$$

$$\gamma_{xy} = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$$

$$= \frac{E(xy) - E(x)\,E(y)}{\sqrt{E(x^2) - (E(x))^2}\ \sqrt{E(y^2) - (E(y))^2}}$$

$$= \frac{4695 - (68 \times 69)}{\sqrt{4628.5 - (68)^2}\ \sqrt{4766.5 - (69)^2}}$$

$$= \frac{3}{(2.121)\,(2.345)}$$

$$\boxed{\gamma_{xy} = 0.603}$$